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# ON $C^*$-ALGEBRAS ASSOCIATED TO THE CONJUGATION REPRESENTATION OF A LOCALLY COMPACT GROUP

EBERHARD KANIUTH AND ANNETTE MARKFORT

ABSTRACT. For a locally compact group $G$, let $\gamma_G$ denote the conjugation representation of $G$ in $L^2(G)$. In this paper we are concerned with nuclearity of $C^*$-algebras associated to $\gamma_G$ and the question of when these are of bounded representation type.

## INTRODUCTION

Let $G$ be a locally compact group with left Haar measure and $C^*(G)$ the group $C^*$-algebra of $G$. For any unitary representation $\pi$ of $G$, there are two $C^*$-algebras associated to $\pi$. The first one is $\pi(C^*(G))$, which henceforth will be denoted $C^*(\pi)$, and the second one is $C^*(\pi(G))$, the $C^*$-algebra generated by the set of operators $\pi(x)$, $x \in G$, on the Hilbert space of $\pi$. If $G_d$ stands for the same group $G$ endowed with the discrete topology and $i_G : G_d \to G$ for the identity, then $C^*(\pi(G)) = C^*(\pi \circ i_G)$. Thus, investigating $C^*(\pi(G))$ naturally involves $G_d$.

For $\pi$ the left regular representation $\lambda_G$ of $G$, $C^*(\lambda_G)$ is called the reduced group $C^*$-algebra which is usually denoted by $C_r^*(G)$. It has been a matter of enormous interest in harmonic analysis and is one of the most important examples in the general theory of $C^*$-algebras. Very recently, Bédos [2] has drawn attention to $C^*(\lambda_G \circ i_G)$ and has shown that amenability of $G$ and of $G_d$ can both be characterized in terms of $C^*(\lambda_G \circ i_G)$.

In this paper we study $C^*$-algebras associated to the conjugation representation $\gamma_G$ of $G$ on $L^2(G)$ which is defined by

$$\gamma_G(x)f(y) = \delta(x)^{1/2} f(x^{-1}yx), \qquad f \in L^2(G),\ x, y \in G,$$

where $\delta$ denotes the modular function of $G$. We show that nuclearity of either $C^*(\gamma_{G_d})$ or $C^*(\gamma_G \circ i_G)$ forces $G_d$ to be amenable (Theorem 1.2). Conversely, if $G_d$ is amenable then $C^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are isomorphic (Theorem 1.7) and nuclear. Unfortunately, in this regard nothing substantial can be said about $C^*(\gamma_G)$ for arbitrary $G$ except that, of course, amenability of $G$ implies that $C^*(\gamma_G)$ is nuclear.

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These results will be applied in §2, where we deal with the question of when any one of the $C^*$-algebras $C^*(\gamma_G)$, $C^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ is of bounded representation type, that is, possesses only finite-dimensional irreducible representations and there is an upper bound for the dimensions. Clearly, since $\gamma_G$ is trivial on $Z(G)$, the centre of $G$, such conditions can only be reflected by the structure of the factor group $G/Z(G)$. It turns out that, for a compactly generated Lie group $G$, any one of the above $C^*$-algebras being of bounded representation type is equivalent to the existence of an abelian subgroup of finite index in $G/Z(G)$ (Theorem 2.10).

The conjugation representation is of interest not least because of its connections to questions on inner invariant means on $L^\infty(G)$ (compare [17], [18] and [13]) and the structure of $G/Z(G)$ [14]. However, so far it is much less understood than the left regular representation. The main difficulty arising is that, even for finite groups, the support of $\gamma_G$ is generally strictly contained in the dual of $G/Z(G)$ and is intricate to determine (compare [11], [12], [13], [20], and [22]).

## Preliminaries and notation

Let $G$ be a locally compact group. We use the same letter, for example $\pi$, for a unitary representation of $G$ and for the corresponding $*$-representation of $C^*(G)$, and $\mathscr{H}(\pi)$ always denotes the Hilbert space of $\pi$. Let $\ker \pi$ be the $C^*$-kernel of $\pi$. If $S$ and $T$ are sets of unitary representations of $G$, then $S$ is *weakly contained* in $T$ $(S \prec T)$ if $\bigcap_{\sigma\in S} \ker\sigma \supseteq \bigcap_{\tau\in T} \ker\tau$ or, equivalently, if any positive definite function associated to $S$ can be uniformly approximated on compact subsets of $G$ by sums of positive definite functions associated to $T$. $S$ and $T$ are *weakly equivalent* $(S \sim T)$ if $S \prec T$ and $T \prec S$. The *dual space* $\widehat{G}$ is the set of equivalence classes of irreducible representations of $G$, endowed with the Jacobson topology. As general references to dual spaces and representation theory we mention [5] and [7].

For any representation of $\pi$ of $G$, the *support* of $\pi$ is the closed subset $\operatorname{supp}\pi = \{\rho \in \widehat{G};\ \rho \prec \pi\}$ of $\widehat{G}$. In particular, the support of the left regular representation $\lambda_G$ is the reduced dual $\widehat{G}_r$.

Recall that amenability of $G$ is equivalent to a number of different conditions: $C^*(\lambda_G) = C^*(G)$, $\widehat{G}_r = \widehat{G}$, or $1_G \prec \lambda_G$, where $1_G$ is the trivial one-dimensional representation of $G$. Concerning amenability we refer to [8], [23] and [24].

Also, we remind the reader that a $C^*$-algebra $A$ is called *nuclear* if there exists exactly one $C^*$-norm on the algebraic tensor product $A \otimes B$ for every $C^*$-algebra $B$. For properties equivalent to nuclearity and a short overview on this concept we refer to [23, §1.31].

Let $N$ be a closed normal subgroup of $G$. Then every representation of $G/N$ can be lifted to a representation of $G$, and in this sense will also be regarded as a representation of $G$. In particular $(G/N)^\frown \subseteq \widehat{G}$. If $H$ is a subgroup of $G$, and $\sigma$ and $\pi$ are representations of $H$ and $G$, respectively, then $\operatorname{ind}_H^G \sigma$ denotes the representation of $G$ induced by $\sigma$ and $\pi \mid H$ the restriction of $\pi$ to $H$. A readable account of the theory of induced representations can be found in [7, Chapter 11]. We will use throughout the fact that inducing and restricting representations are continuous with respect to Fell's topology [6].

Next, let

$$\{e\} = Z_0(G) \subseteq Z(G) = Z_1(G) \subseteq Z_2(G) \subseteq \cdots$$

be the ascending central series and $G_f$ the finite conjugacy class subgroup of $G$. For any two subsets $M$, $N$ of $G$ we denote by $C_M(N)$ the centralizer of $N$ in $M$. If $M = G$ we often omit the index. Using this notation, for discrete groups $G$, $\gamma_G$ is weakly equivalent to the set $\{\mathrm{ind}_{C(a)}^G 1_{C(a)};\, a \in G\}$ (see [13, p. 27]).

For general $G$ the only available description of $\operatorname{supp}\gamma_G$ is as follows. Let $G$ be a $\sigma$-compact locally compact group, and suppose that $C^*(\lambda_G)$ is nuclear. Then by [11, Theorem]

$$\operatorname{supp}\gamma_G = \overline{\bigcup_{\pi \in \widehat{G}_r} \operatorname{supp}(\pi \otimes \bar{\pi})}.$$

## 1. $C^*(\gamma_{G_d})$, $C^*(\gamma_G \circ i_G)$, AND AMENABILITY

We start with a lemma which will be used in the proof of Theorem 1.2 below as well as in §2.

**Lemma 1.1.** *For any locally compact group $G$ and $i_G : G_d \to G$ the identity*

$$\lambda_{G_d/(G_d)_f} \prec \gamma_G \circ i_G.$$

*Proof.* The proof is an adaptation of the proof of Theorem 1.8 in [13]. Let $D = G_d$ and recall that $\lambda_{D/D_f}$ is the GNS-representation defined by the characteristic function $\chi_{D_f}$ of $D_f$. Therefore it suffices to show that given any finite subset $F$ of $D$, there exists a positive definite function $\varphi$ associated to $\gamma_G \circ i_G$ such that $\varphi \,|\, F = \chi_{D_f} \,|\, F$. Set $F_1 = F \cap D_f$ and $F_2 = F \backslash F_1$. Then, by the proof of [13, Theorem 1.8], there exists $a \in C(F_1)$ such that $x^{-1}ax \neq a$ for all $x \in F_2$.

$C(F_1)$ is a closed subgroup of finite index in $G$, and hence is open. Thus we find an open neighbourhood $V$ of $a$ in $G$ such that $V \subseteq C(F_1)$ and $x^{-1}Vx \cap V = \varnothing$ for all $x \in F_2$. Observe that $\delta(x) = 1$ for all $x \in F_1$ since $x^{-1}Vx = V$. Now, let $f = |V|^{-1/2}\chi_V$ and

$$\varphi(x) = \langle \gamma_G(x) f, f \rangle = \delta(x)^{1/2} |V|^{-1} \int_V \chi_V(x^{-1}yx)\, dy.$$

It follows that $\varphi(x) = 1$ for $x \in F_1$ as $V \subseteq C(F_1)$, and $\varphi(x) = 0$ for $x \in F_2$ since $x^{-1}Vx \cap V = \varnothing$ for $x \in F_2$. □

**Theorem 1.2.** *For a locally compact group $G$ the following are equivalent.*

(i) *$G_d$ is amenable.*

(ii) *$C^*(\gamma_G \circ i_G)$ is nuclear.*

(iii) *$C^*(\gamma_{G_d})$ is nuclear.*

*Proof.* (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious, since amenability of $G_d$ implies that $C^*(G_d)$ is nuclear, and hence so are the quotients $C^*(\gamma_G \circ i_G)$ and $C^*(\gamma_{G_d})$ of $C^*(G_d)$ (compare [4, Corollary 4]).

Since $\lambda_{G_d/(G_d)_f} \prec \gamma_G \circ i_G$ (Lemma 1.1), $C^*(\lambda_{G_d/(G_d)_f})$ is a quotient of $C^*(\gamma_G \circ i_G)$. Thus (ii) implies nuclearity of $C^*(\lambda_{G_d/(G_d)_f})$, and by [16, Theorem 4.2] this forces $G_d/(G_d)_f$ to be amenable. Now groups with finite conjugacy classes are well known to be amenable (see [24, Proposition 12.9 or Corollary

14.26]). As the class of amenable groups is closed under forming extensions by amenable groups, $G_d$ turns out to be amenable. (iii) $\Rightarrow$ (i) follows in the same way by appealing to Theorem 1.8 of [13] instead of Lemma 1.1 $\square$

For $\gamma_G$ replaced by the left regular representation, Theorem 1.2 has been established in [2, Theorem 3].

**Lemma 1.3.** *Let $G$ and $H$ be locally compact groups, and let $j : H \to G$ be a continuous and injective homomorphism with dense range. Then $\widehat{G} \circ j \subseteq \widehat{H}$, and $\widehat{G} \circ j$ is dense in $\widehat{H}$ provided that $H$ is discrete and amenable.*

*Proof.* Let $\pi_1$, $\pi_2$ be representations of $G$. If $\pi_1$ and $\pi_2$ are equivalent, then $\pi_1 \circ j$ and $\pi_2 \circ j$ are equivalent representations of $H$. Conversely, if $\pi_1 \circ j$ and $\pi_2 \circ j$ are equivalent, then since $j(H)$ is dense in $G$ and representations are strongly continuous, it follows immediately that $\pi_1$ and $\pi_2$ are equivalent. Moreover, for a representation $\pi$ of $G$, $\pi$ is irreducible if and only if $\pi \circ j$ is irreducible. Thus $\pi \to \pi \circ j$ induces an injective mapping from $\widehat{G}$ into $\widehat{H}$.

It is easy to see that the Dirac function $\delta_e$ on $H$ can be pointwise approximated by positive definite functions associated to $\lambda_G \circ j$ [3, Proposition 1]. For $H$ discrete, this shows that $\lambda_H \prec \lambda_G \circ j$, and hence $\widehat{G} \circ j$ is dense in $\widehat{H}$ if, in addition, $H$ is amenable. $\square$

**Corollary 1.4.** *Suppose that $H$ is amenable and discrete, and let $G$ and $j$ be as in Lemma 1.3. Then*

$$\{(\pi \circ j) \otimes (\bar{\pi} \circ j) ; \pi \in \widehat{G}\} \sim \{\rho \otimes \bar{\rho} ; \rho \in \widehat{H}\}.$$

*Proof.* Let $P$ and $R$ denote the set of representations on the left and on the right, respectively. It is clear from $\widehat{G} \circ j \subseteq \widehat{H}$ that $P \prec R$. On the other hand, since $\widehat{G} \circ j$ is dense in $\widehat{H}$ by Lemma 1.3, for $\rho \in \widehat{H}$ every coordinate function of the form

$$x \to \langle (\rho \otimes \bar{\rho})(x)(\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2 \rangle = \langle \rho(x)\xi_1, \eta_1 \rangle \langle \bar{\rho}(x)\xi_2, \eta_2 \rangle,$$

where $\xi_1, \eta_1 \in \mathscr{H}(\rho)$ and $\xi_2, \eta_2 \in \mathscr{H}(\bar{\rho})$, can be approximated on finite subsets of $H$ by a product of functions each of which is a finite sum of positive definite functions associated to $\pi \circ j$ and $\bar{\pi} \circ j$, $\pi \in \widehat{G}$, respectively. It follows that $\rho \otimes \bar{\rho} \prec P$. $\square$

We have to compare $\gamma_{G_d}$ and $\gamma_G \circ i_G$ with respect to weak equivalence. As mentioned in the proof of Lemma 1.3, for every locally compact group $G$, $\lambda_{G_d} \prec \lambda_G \circ i_G$. In general, however, $\gamma_{G_d}$ need not be weakly contained in $\gamma_G \circ i_G$. We will further comment on this in Lemma 1.8 and Remarks 1.9. But at least we have

**Corollary 1.5.** *Suppose that $G$ is $\sigma$-compact and $H$ is amenable and discrete, and let $j$ be as in Lemma 1.3. Then $\gamma_H \prec \gamma_G \circ j$.*

*Proof.* Since $G$ is amenable and $\sigma$-compact, $\gamma_G \sim \{\pi \otimes \bar{\pi} ; \pi \in \widehat{G}\}$ by the theorem of [11]. Corollary 1.4 yields

$$\gamma_G \circ j \sim \{(\pi \circ j) \otimes (\bar{\pi} \circ j) ; \pi \in \widehat{G}\} \sim \{\rho \otimes \bar{\rho} ; \rho \in \widehat{H}\},$$

and this latter set weakly contains $\gamma_H$ [11, Corollary 1]. $\square$

**Lemma 1.6.** *Let $G$ be a second countable group such that $G_d$ is amenable. Then* $\gamma_G \circ i_G \prec \gamma_{G_d}$.

*Proof*. There exists a countable dense subset in $G$ as $G$ is second countable. Thus every finite subset of $G$ is contained in some countable dense subgroup $H$ of $G$. For any such $H$, $\{\rho \otimes \bar{\rho};\, \rho \in \widehat{H}_d\} \sim \gamma_{H_d}$, and hence by Corollary 1.4,

$$\gamma_G \circ j_H \sim \{(\pi \circ j_H) \otimes (\bar{\pi} \circ j_H);\, \pi \in \widehat{G}\} \sim \{\rho \otimes \bar{\rho};\, \rho \in \widehat{H}_d\} \sim \gamma_{H_d},$$

where $j_H$ denotes the inclusion $H_d \to G$. On the other hand, $\gamma_{H_d}$ is a subrepresentation of $\gamma_{G_d} \mid H_d$ and

$$\langle \gamma_G \circ i_G(x) f, f\rangle = \langle \gamma_G \circ j_H(x) f, f\rangle$$

for all $x \in H$ and $f \in L^2(G)$. This proves $\gamma_G \circ i_G \prec \gamma_{G_d}$. □

**Theorem 1.7.** *Let $G$ be a locally compact group. If $G_d$ is amenable, then $\gamma_G \circ i_G \sim \gamma_{G_d}$, and $C^*(\gamma_G \circ i_G)$ and $C^*(\gamma_{G_d})$ are isomorphic.*

*Proof*. We first reduce to the $\sigma$-compact case. To that end, let $\mathfrak{H}$ denote the set of all $\sigma$-compact open subgroups $H$ of $G$, and suppose that we already know $\gamma_H \circ i_H \sim \gamma_{H_d}$ for every $H \in \mathfrak{H}$. To show that $\gamma_G \circ i_G \prec \gamma_{G_d}$, let a finite subset $F$ of $G$ and $f \in L^2(G)$ be given and consider the function $\varphi(x) = \langle \gamma_G(x) f, f\rangle$. Choose $H \in \mathfrak{H}$ such that $F \subseteq H$ and $f \mid G\backslash H = 0$. Then $\varphi(x) = \langle \gamma_H(x) f \mid H, f \mid H\rangle$ for all $x \in H$. Since $\gamma_H \circ i_H \prec \gamma_{H_d}$, $\varphi$ can be approximated on $F$ by sums of positive definite functions associated to $\gamma_{H_d}$. It follows that

$$\gamma_G \circ i_H \prec \gamma_{H_d} \prec \gamma_{G_d} \mid H_d,$$

and hence $\gamma_G \circ i_G \prec \gamma_{G_d}$. That, conversely, $\gamma_{G_d} \prec \gamma_G \circ i_G$ is seen in the same way.

Recall next that, by Corollary 1.5, $\gamma_H \circ i_H \succ \gamma_{H_d}$ for each $H \in \mathfrak{H}$. From Lemma 1.6 we know that conversely $\gamma_H \circ i_H \prec \gamma_{H_d}$ provided that $H$ is second countable. Thus it remains to extend this to the case of a $\sigma$-compact group $H$.

Being $\sigma$-compact, $H$ is a projective limit of second countable groups $H_\alpha = H/K_\alpha$, $\alpha \in A$, where the $K_\alpha$ are compact. Now, the set

$$\{f \in C_c(H);\ \text{for some } \alpha \in A,\ f(xk) = f(x) \text{ for all } x \in H \text{ and } k \in K_\alpha\}$$

is dense in $C_c(H)$ in the inductive limit topology. Therefore it suffices to approximate a function $x \to \langle \gamma_H(x) f, f\rangle$, where $f \in C_c(H)$ is constant on cosets of some $K = K_\alpha$, on finite subsets of $H$ by sums of positive definite functions associated to $\gamma_{H_d}$. Define $g$ on $H/K$ by $g(xK) = f(x)$ for $x \in H$. Then

$$\langle \gamma_H(x) f, f\rangle = \langle \gamma_{H/K}(xK) g, g\rangle,$$

and by Lemma 1.6 the function on the right can be approximated on finite subsets of $H/K$ by sums of positive definite functions associated to $\gamma_{(H/K)_d}$. Now, $(H/K)_d = H_d/K_d$, and by [20, Lemma 1.1], $\gamma_{H_d/K_d} \prec \gamma_{H_d}$ since $H_d$ is amenable. This shows that $\gamma_H \circ i_H \prec \gamma_{H_d}$ and finishes the proof. □

Obviously, if $G_d$ is amenable, then so is $G$. As to the regular representation, it has been observed in [2, Theorem 3] that if $G$ is amenable and $\lambda_{G_d} \sim \lambda_G \circ i_G$, then $G_d$ is amenable. In fact, under these assumptions,

$$1_{G_d} = 1_G \circ i_G \prec \lambda_G \circ i_G \sim \lambda_{G_d}.$$

Although it is conceivable, we do not know whether, as a converse to Theorem 1.7, amenability of $G$ and $\gamma_{G_d} \sim \gamma_G \circ i_G$ imply that $G_d$ is amenable.

We conclude this section by returning to the question of when $\gamma_{G_d} \prec \gamma_G \circ i_G$. Recall that a locally compact group is said to be an [SIN]-group if $G$ has a system of neighbourhoods $V$ of the identity such that $x^{-1}Vx = V$ for all $x \in G$.

**Lemma 1.8.** *If $G$ is an [SIN]-group, then* $\gamma_{G_d} \prec \gamma_G \circ i_G$.

*Proof.* It suffices to approximate the function $x \to \chi_{C(a)}(x) = \langle \gamma_{G_d}(x)\delta_a, \delta_a \rangle$, $a \in G$, on finite subsets $F$ of $G$ by positive definite functions associated to $\gamma_G \circ i_G$. Now, given such an $F$, there exists an invariant symmetric neighbourhood $V$ of $e$ in $G$ such that $x^{-1}ax \notin V^2 a$ for all $x \in F \backslash C(a)$. Let $\varphi = |V|^{-1/2}\chi_{V_a}$; then it is easily verified that

$$\langle \gamma_G(x)\varphi, \varphi \rangle = |V|^{-1} \int_V \chi_{V_a}(x^{-1}vax)\, dv$$

is equal to 1 for all $x \in C(a)$ and equal to 0 for all $x \in F\backslash C(a)$. □

*Remarks* 1.9. (i) Suppose that $C^*(\lambda_G)$ is nuclear and that $\gamma_{G_d} \prec \gamma_G \circ i_G$. Then $G$ is amenable. This can be seen as follows. Since

$$1_{G_d} \prec \gamma_{G_d} \prec \gamma_G \circ i_G \prec \lambda_G \circ i_G$$

[11, Proposition 1], there is a homomorphism of $C^*(\lambda_G(G)) = C^*(\lambda_G \circ i_G)$ onto $\mathbb{C}$. By [2, Theorem 1] this implies that $G$ is amenable. In particular, for any noncompact connected semisimple Lie group $G$, $\gamma_{G_d}$ is not weakly contained in $\gamma_G \circ i_G$.

(ii) By Lemma 1.8 for $G$ compact, $\gamma_{G_d} \prec \gamma_G \circ i_G$. Moskowitz [22] has shown that, for $G$ a compact connected Lie group, $\operatorname{supp} \gamma_G = (G/Z(G))^\wedge$. This can be used to compare the sets $\operatorname{supp}(\gamma_G \circ i_G)$, $(\operatorname{supp} \gamma_G) \circ i_G$, and $\operatorname{supp} \gamma_{G_d}$. As an illustrating example let us look at $G = SO(3)$. Then $(\operatorname{supp} \gamma_G) \circ i_G = \widehat{G} \circ i_G$, and $\widehat{G} \circ i_G$ fails to be dense in $\widehat{G}_r$ (see [3, Corollary 1]).

Considering $G_d$, it follows from [13, Corollary 1.9] that $\operatorname{supp} \gamma_{G_d} = (G_d)_r^\wedge \cup \{1_{G_d}\}$ since $(G_d)_f$ is trivial and the centralizer of each matrix in $SO(3)\backslash\{E\}$ has a subgroup of index 2, which is conjugate to $SO(2)$. Thus $\operatorname{supp} \gamma_{G_d} \cap (\operatorname{supp} \gamma_G) \circ i_G = \{1_{G_d}\}$ and $\operatorname{supp} \gamma_{G_d}$ is strictly contained in $\operatorname{supp}(\gamma_G \circ i_G)$, since $1_{G_d}$ is the only finite-dimensional representation in $\operatorname{supp} \gamma_{G_d}$.

## 2. When is $C^*(\gamma_G)$ of bounded representation type?

Let $A$ be a $C^*$-algebra and $\widehat{A}$ its dual space. $A$ is said to be of *bounded representation type* (b.r.t.) if every $\pi \in \widehat{A}$ is finite dimensional and if there is an upper bound for these dimensions. The analogous notion applies to representations. Thus, a representation $\rho$ of $A$ is of b.r.t. if $\rho(A)$ is of b.r.t. Moreover, a locally compact group $G$ is of bounded representation type if $C^*(G)$ has this property. The first paper dealing with such groups that we are aware of is [15]. Groups of b.r.t. have finally been identified by Moore [21] as precisely those which have an abelian subgroup of finite index.

In this section we are interested in the question of when the $C^*$-algebras $C^*(\gamma_G)$, $C^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are of bounded representation type. For such a particular representation, this appears to be a rather intricate problem. We succeeded in resolving it for compactly generated Lie groups, where by Lie group we mean a locally compact group $G$ whose connected component $G_0$ of the identity is open and is an analytic group. However, we were unable to characterize non-finitely-generated discrete groups $G$ or totally disconnected compact groups $G$ with $C^*(\gamma_G)$ of b.r.t.

It is worth commenting here on the same question for the left regular representation. Now, for any locally compact group $H$, $C^*(\lambda_H)$ being of b.r.t. implies that $H$ has an abelian subgroup of finite index. Indeed, this follows from [26, Satz 2] and can also be deduced from Moore's results [21]. As to $C^*(\lambda_H \circ i_H)$, notice that by [2, Lemma 2] $\lambda_{H_d}$ is weakly contained in $\lambda_H \circ i_H$, so that $C^*(\lambda_{H_d})$ is of b.r.t. provided that $C^*(\lambda_H \circ i_H)$ is.

*Remarks* 2.1. (i) If $\gamma_G$ is of bounded representation type (b.r.t.), then $\gamma_G \,|\, H$ is of b.r.t. for every closed subgroup $H$ of $G$. Indeed, let

$$T = \bigcup_{\pi \in \operatorname{supp} \gamma_G} \operatorname{supp}(\pi \,|\, H) \subseteq \widehat{H},$$

and suppose that $\dim \pi \leq d$ for all $\pi \in \operatorname{supp} \gamma_G$. Then $\dim \tau \leq d$ for all $\tau \in T$, and hence for all $\tau \in \overline{T}$. On the other hand, $\overline{T} = \operatorname{supp}(\gamma_G \,|\, H)$ since $T$ is weakly equivalent to $\gamma_G \,|\, H$.

(ii) Let $H$ be an open subgroup of $G$. If $\gamma_G$ is of b.r.t., then so is $\gamma_H$. In fact, by (i) $\gamma_G \,|\, H$ is of b.r.t., and $\gamma_H$ is a subrepresentation of $\gamma_G \,|\, H$ as $L^2(H)$ is a subspace of $L^2(G)$. Notice, however, that in general for a closed subgroup $H$ of $G$, $\gamma_H$ need not even be weakly contained in $\gamma_G \,|\, H$ (see [14]).

(iii) If $\gamma_G$ is of b.r.t. and $C^*(\lambda_G)$ is nuclear, then $G$ is amenable. The nuclearity assumption guarantees that $\gamma_G \prec \lambda_G$ [11, Proposition 1]. Now, it is well known that $G$ is amenable provided that $\lambda_G$ weakly contains a finite-dimensional representation. Recall that $C^*(\lambda_G)$ ($C^*(G)$, as a matter of fact) is nuclear if $G/G_0$ is amenable.

If $N$ is a closed normal subgroup of $G$, then $G$ acts on $\widehat{N}$ by $(x, \lambda) \to \lambda^x$, where $\lambda^x(n) = \lambda(x^{-1}nx)$ for $x \in G$ and $n \in N$, and $G_\lambda$ denotes the stability subgroup of $\lambda$ in $G$ under this action.

**Lemma 2.2.** *Let $G$ be a locally compact group, and suppose that $\operatorname{supp} \gamma_G$ contains a dense subset of finite-dimensional representations. Let $N$ be a closed normal subgroup of $G$ such that $N/N \cap Z(G)$ is a vector group. Then there exists a closed subgroup $H$ of finite index in $G$ such that $N \subseteq Z_2(H)$.*

*Proof.* Let $\Pi = \{\pi \in \operatorname{supp} \gamma_G;\, \dim \pi < \infty\}$ and $\Lambda = \bigcup_{\pi \in \Pi} \operatorname{supp}(\pi \,|\, N)$. By hypothesis, $\gamma_G \,|\, N \sim \Pi \,|\, N \sim \Lambda$, so that $\Lambda$ separates the points of $V = N/N \cap Z(G)$. $V$ and hence $\widehat{V}$ being a vector group, $\Lambda$ contains a basis $\{\lambda_1, \dots, \lambda_m\}$ of $\widehat{V}$. Now, $H = \bigcap_{j=1}^m G_{\lambda_j}$ has finite index in $G$ and $\lambda_j^h = \lambda_j$ for all $h \in H$ and $1 \leq j \leq m$. Since continuous automorphisms of vector groups are linear, it follows that $\lambda^h = \lambda$ for all $\lambda \in \widehat{V}$ and $h \in H$. This implies that $N/N \cap Z(G) \subseteq Z(H/N \cap Z(G))$ and hence $N \subseteq Z_2(H)$. □

**Lemma 2.3.** *Let $G$ and $\gamma_G$ be as in Lemma 2.2. Let $N$ be a closed normal subgroup of $G$ such that $N/N \cap Z(G) = \mathbb{T}^m$ for some $m \in \mathbb{N}$. Then $N \subseteq Z_2(H)$ for some subgroup $H$ of finite index in $G$.*

*Proof.* Let $\Pi$ and $\Lambda$ be as in the proof of the previous lemma. Then $\Lambda$ generates $(N/N \cap Z(G))^\wedge = \mathbb{Z}^m$, so that $G_\lambda$ has finite index in $G$ for each $\lambda \in \mathbb{Z}^m$. As $\mathbb{Z}^m$ is finitely generated, we find a subgroup $H$ of finite index in $G$ such that $\lambda^h = \lambda$ for all $\lambda \in \mathbb{Z}^m$ and all $h \in H$. This proves that $N/N \cap Z(G) \subseteq Z(H/N \cap Z(G))$ and hence $N \subseteq Z_2(H)$. $\square$

**Lemma 2.4.** *Let $K$ be a compact connected normal subgroup of the Lie group $G$. If $\gamma_G$ is of b.r.t., then the commutator subgroup $K'$ of $K$ is contained in the centre of $G$.*

*Proof.* It suffices to show that $K' \subseteq Z(H)$ for every $\sigma$-compact open subgroup $H$ of $G$. Since $\gamma_H$ is of b.r.t. for every such $H$, we can assume that $G$ is $\sigma$-compact and hence second countable as it is a Lie group. Recall that by [19, Lemma 3.1], for any second countable group $G$, $\gamma_G$ is unitarily equivalent to the restriction of $\mathrm{ind}_{\Delta_G}^{G\times G} 1_{\Delta_G}$ to $\Delta_G$ where $\Delta_G$ denotes the diagonal subgroup of $G \times G$. Since $K$ is compact and $G$ is second countable, $\Delta_K$ and $\Delta_G$ are regularly related in $G \times G$ in the sense of Mackey. Therefore, by [6, Theorem 5.3], with $\Delta_G^u = u\Delta_G u^{-1}$ for $u \in G \times G$,

$$\begin{aligned}\gamma_G \,|\, K = \mathrm{ind}_{\Delta_G}^{G\times G} 1_{\Delta_G} \,|\, \Delta_K &\sim \{\mathrm{ind}_{u^{-1}\Delta_G u \cap \Delta_K}^{\Delta_K} 1_{u^{-1}\Delta_G u\cap\Delta_K} \,;\, u \in G \times G\} \\ &= \{\mathrm{ind}_{C(a)\cap K}^{K} 1_{C(a)\cap K} \,;\, a \in G\} = \{\mathrm{ind}_{C_K(a)}^{K} \,;\, a \in G\}.\end{aligned}$$

Fix $a \in G$, and let $N(a)$ denote the greatest normal subgroup of $K$ contained in $C_K(a)$. There exist finitely many $x_1, \dots, x_m \in K$ such that

$$N(a) = \bigcap_{j=1}^{m} x_j^{-1} C_K(a) x_j$$

(compare [1, Proposition 2.1]). By [6, Theorem 5.5] the $m$-fold tensor product $(\gamma_G \,|\, K)^{\otimes m}$ weakly contains

$$\mathrm{ind}^K_{x_1^{-1}C_K(a)x_1\cap\cdots\cap x_m^{-1}C_K(a)x_m} 1_{x_1^{-1}C_K(a)x_1\cap\cdots\cap x_m^{-1}C_K(a)x_m} = \mathrm{ind}_{N(a)}^K 1_{N(a)}.$$

Now tensor products of representations of b.r.t. are again of b.r.t. [25, Lemma 5]. Thus $\mathrm{ind}_{N(a)}^K 1_{N(a)}$ is of b.r.t., and since $K$ is connected this yields that $K/N(a)$ is abelian. It follows that

$$K\Big/\left(\bigcap_{a\in G}(C(a)\cap K)\right) = K\Big/\bigcap_{a\in G} N(a)$$

is abelian. This proves $K' \subseteq \bigcap_{a\in G} C(a) = Z(G)$. $\square$

**Proposition 2.5.** *Let $G$ be a Lie group and $N$ a connected closed normal subgroup of $G$. If $C^*(\gamma_G)$ is of b.r.t., then there exists a subgroup $H$ of finite index in $G$ such that $N \subseteq Z_6(H)$.*

*Proof.* Let $M = N \cap Z(G)$. Since $\gamma_G \,|\, N$ separates the points of $N/M$, $N/M$ is a maximally almost periodic connected Lie group. By the Freudenthal-Weil theorem [5, Théorème 16.4.6] $N/M$ is a direct product of a vector group $W$ and a compact connected Lie group $K$.

Let $q : G \to G/M$ be the quotient homomorphism. As $K$ is normal in $G/M$, it follows from Lemma 2.4 that $K' \subseteq Z(G/M)$ and hence $q^{-1}(K') \subseteq$

$Z_2(G)$. Applying Lemma 1.1 in [14] twice gives $\gamma_{G/q^{-1}(K')} \prec \gamma_G$, so that $\gamma_{G/q^{-1}(K')}$ is of b.r.t. Now, $K/K'$ is a normal torus in $G/q^{-1}(K')$. It follows from Lemma 2.3 that $K/K' \subseteq Z_2(H_1/q^{-1}(K'))$ for some subgroup $H_1$ of finite index in $G$. Thus $q^{-1}(K) \subseteq Z_4(H_1)$.

Now, moving to $G/q^{-1}(K)$, similar arguments apply to the normal vector subgroup $W$ of $G/q^{-1}(K)$. Again, since continuous automorphisms of vector groups are linear, $W \cap Z(G/q^{-1}(K))$ is a vector group and hence so is $W/W \cap Z(G/q^{-1}(K))$. Lemma 2.2 yields that $W \subseteq Z_2(H_2/q^{-1}(K))$ for some subgroup $H_2$ of finite index in $G$ containing $q^{-1}(K)$. With $H = H_1 \cap H_2$, we obtain that $N \subseteq Z_6(H)$. □

*Remark* 2.6. Let $D$ be a discrete group with $\gamma_D$ of b.r.t. Then, since $\lambda_{D/D_f} \prec \gamma_D$ [13, Theorem 1.8], $\lambda_{D/D_f}$ is of b.r.t. and therefore $D/D_f$ has an abelian subgroup of finite index (compare [26, Satz 1]). In particular, $D$ is amenable. It is worthwhile to remind the reader that in order to conclude that a discrete group $G$ is almost abelian it is only required that $\lambda_G$ is of type I [10].

**Corollary 2.7.** *If $G$ is a Lie group with $C^*(\gamma_G)$ of b.r.t., then $G_d$ is amenable and $C^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are both of b.r.t.*

*Proof*. By Proposition 2.5, $G_0 \subseteq Z_m(H)$ for some $m \in \mathbb{N}$ and some subgroup $H$ in $G$ of finite index. In particular, $G_0$ is nilpotent. Let $D = G/G_0$; then repeated application of [14, Lemma 1.1] gives $\gamma_D \prec \gamma_G$. Thus $\gamma_D$ is of b.r.t., and hence $D$ is amenable (Remark 2.6). Since $(G_0)_d$ and $G/G_0$ are amenable, $G_d$ is amenable.

By what we have seen in Theorem 1.7, $\gamma_{G_d} \sim \gamma_G \circ i_G$, and $G^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are isomorphic. Thus it remains to recognize that $\gamma_{G_d}$ is of b.r.t. But this follows because $\gamma_G$ is of b.r.t. and $\operatorname{supp} \gamma_{G_d}$ is contained in the closure of $(\operatorname{supp} \gamma_G) \circ i_G$ in $\widehat{G}_d$. □

**Corollary 2.8.** *For a connected group $G$, $C^*(\gamma_G)$ is of bounded representation type if and only if $G$ is 2-step nilpotent.*

*Proof*. Clearly, if $G/Z(G)$ is abelian, then every $\pi \in \operatorname{supp} \gamma_G$ is one-dimensional. Conversely, suppose that $G$ is connected and $\gamma_G$ is of b.r.t. Then $G$ is a projective limit of Lie groups $G_\iota = G/K_\iota$, $\iota \in I$, where the $K_\iota$ are compact, and every $\gamma_{G_\iota}$ is of b.r.t. Let $q_\iota : G \to G_\iota$ denote the quotient homomorphism. Since $Z(G) = \bigcap_{\iota \in I} q_\iota^{-1}(Z(G_\iota))$, $G$ is 2-step nilpotent if all $G_\iota$ are. Therefore we can assume that $G$ is a Lie group.

By Corollary 2.7, $\gamma_{G_d}$ is of b.r.t., and hence $G/G_f$ has an abelian subgroup of finite index. For any $x \in G_f$, $C(x)$ is a closed subgroup of finite index in $G$, so that $x \in Z(G)$. It follows that $\overline{G}_f \subseteq Z(G)$, and $G/\overline{G}_f$ has a closed abelian subgroup of finite index. $G$ being connected, we obtain that $G/Z(G)$ is abelian. □

**Lemma 2.9.** *Let $D$ be a discrete group such that $\gamma_D$ is of b.r.t. For $x \in D$ let $N(x)$ denote the greatest normal subgroup of $D$ contained in $C(x)$. Suppose that for some finite subset $F$ of $D$, $\bigcap_{x \in F} N(x) = Z(D)$. Then $D/Z(D)$ has an abelian subgroup of finite index.*

*Proof*. Since $\operatorname{ind}_{C(x)}^{D} 1_{C(x)} \prec \gamma_D$ for each $x \in D$, all these quasi-regular representations are of b.r.t. The kernel of $\operatorname{ind}_{C(x)}^{D} 1_{C(x)}$ is $N(x)$ as is easily verified. Now, $\operatorname{ind}_{C(x)}^{D} 1_{C(x)}$ being of b.r.t. is equivalent to the algebra generated by the operators $\operatorname{ind}_{C(x)}^{D} 1_{C(x)}(y)$, $y \in D$, on $l^2(D/C(x))$ satisfying a standard polynomial identity (see [15] and [21]).

Therefore, by Satz 1 of [26], the factor group $D/N(x)$, which is isomorphic to $\operatorname{ind}_{C(x)}^{D} 1_{C(x)}(D)$, has an abelian subgroup $A(x)/N(x)$ of finite index. With

$$A = \bigcap_{x\in F} A(x)$$

it follows that $A$ has finite index in $D$ and

$$A' \subseteq \bigcap_{x\in F} A(x)' \subseteq \bigcap_{x\in F} N(x) = Z(D). \quad \square$$

**Theorem 2.10.** *For a compactly generated Lie group $G$ the following conditions are equivalent*:

(i) $C^*(\gamma_G)$ *is of bounded representation type.*
(ii) $C^*(\gamma_{G_d})$ *is of bounded representation type.*
(iii) $C^*(\gamma_G \circ i_G)$ *is of bounded representation type.*
(iv) $G/Z(G)$ *possesses an abelian subgroup of finite index.*

*Proof*. (iv) $\Rightarrow$ (i), (ii), (iii) are clear since all three representations $\gamma_G$, $\gamma_{G_d}$, and $\gamma_G \circ i_G$ are trivial on $Z(G) = Z(G_d)$. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are consequences of Corollary 2.7.

Notice next that (iii) $\Rightarrow$ (ii). In fact, if $C^*(\gamma_G \circ i_G)$ is of b.r.t., then so is $C^*(\lambda_{G_d/(G_d)_f})$ by Lemma 1.1. This implies that $G_d/(G_d)_f$ is almost abelian and hence $G_d$ is amenable. Theorem 1.8 now shows that $C^*(\gamma_{G_d})$ is of b.r.t.

It remains to show (ii) $\Rightarrow$ (iv). For that we want to apply Lemma 2.9. Thus we have to produce a finite subset $F$ of $G$ such that $\bigcap_{x\in F} N(x) = Z(G)$.

To construct $F$ let $Z_0 = Z(G) \cap G_0$ and notice that $\gamma_G \mid G_0$ separates the points of $G_0/Z_0$ and is of b.r.t. by Remarks 2.1 (i). Therefore $G_0/Z_0$ is a maximally almost periodic connected Lie group. It follows from the Freudenthal-Weil theorem (see [5, Théorème 16.4.6]) that $G_0/Z_0$ is a direct product of a compact Lie group $K$ and some $\mathbb{R}^m$. Now, $\gamma_{G/Z_0}$ is of b.r.t. and $K$ is normal in $G/Z_0$. An application of Lemma 2.4 yields that $K$ is 2-step nilpotent. As is well known this implies that $K$, being a compact connected Lie group, is a torus $\mathbb{T}^n$.

Let $q : G \to G/G_0$ and $h : G \to G/Z_0$ denote the quotient homomorphisms. Choose a finite subset $F_1$ of $G$ such that $q(F_1)$ generates $G/G_0$ as a group. Both $\mathbb{R}^m$ and $\mathbb{T}^n$ contain finitely generated dense subgroups. Thus there exist finite subsets $F_2$ and $F_3$ of $G_0$ such that $h(F_2)$ and $h(F_3)$ generate a dense subgroup of $\mathbb{R}^m$ and $\mathbb{T}^n$, respectively. Finally, let $F = F_1 \cup F_2 \cup F_3$. It is now obvious that $F \cup Z_0$ generates a dense subgroup of $G$, whence $C(F) = Z(G)$. This completes the proof. $\square$

One might well expect that Theorem 2.10 remains true when the assumption that $G$ be compactly generated is dropped. However, as mentioned earlier, we did not succeed in proving that if $G$ is a (not necessarily finitely generated) discrete group with $C^*(\gamma_G)$ of b.r.t., then $G/Z(G)$ must be almost abelian. This is surprising since we already know that $G/G_f$ is almost abelian (Remark 2.6). The point is that it seems to be difficult to handle discrete groups with finite conjugacy classes (so-called [FC]-groups). We finish the paper by looking at a special class of [FC]-groups.

**Example 2.11.** Let $G$ be the restricted direct product of finite groups $G_\iota$, $\iota \in I$. We claim that the following conditions are equivalent:

(i) $C^*(\gamma_G)$ is of bounded representation type.

(ii) $\dim \pi < \infty$ for every $\pi \in \operatorname{supp} \gamma_G$.

(iii) $G_\iota$ is 2-step nilpotent for almost all $\iota \in I$.

Condition (iii) implies that $G/Z_2(G)$ is finite, whence (i) follows. To verify (ii) $\Rightarrow$ (iii), first consider a finite group $F$. Suppose that $\operatorname{supp}(\sigma \otimes \bar{\sigma}) \subseteq (F/F')^\frown$ for all $\sigma \in \widehat{F}$. Then $\sigma \mid F'$ has to be a multiple of a $G$-invariant character for all $\sigma \in \widehat{F}$, and this yields $F' \subseteq Z(F)$. Thus, if $F$ fails to be 2-step nilpotent, then for at least one $\sigma \in \widehat{F}$, $\sigma \otimes \bar{\sigma}$ has an irreducible subrepresentation of dimension $\geq 2$.

Now, suppose that $G_\iota$ is not 2-step nilpotent for all $\iota$ in some infinite subset $J$ of $I$. For each $\iota \in J$, choose $\sigma_\iota \in \widehat{G}_\iota$ and $\tau_\iota \in \operatorname{supp}(\sigma_\iota \otimes \bar{\sigma}_\iota)$ with $\dim \tau_\iota \geq 2$. For $\iota \in I \backslash J$, let $\sigma_\iota = \tau_\iota = 1_{G_\iota}$. The infinite tensor products $\pi = \bigotimes_{\iota \in I} \sigma_\iota$ and $\rho = \bigotimes_{\iota \in I} \tau_\iota$ are irreducible [9, §11], $\rho$ is infinite dimensional, and $\rho \in \operatorname{supp}(\pi \otimes \bar{\pi})$. This contradicts (ii).

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Fachbereich Mathematik/Informatik, Universität Paderborn, D-33095 Paderborn, Germany

*E-mail address*: kaniuth@uni-paderborn.de